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We find a new description of curves that trivializes arc length computations. For any given a curve in space, there are many different vector-valued functions thatdraw this curve. For example, consider a circle of radius centered at theorigin. Each of the following vector-valued functions will draw this circle: Each of these functions is a different parameterization of the circle. This means thatwhile these vector-valued functions draw the same circle, they do so at differentrates. Considering , , and , which draws the circle of radius quickest? Which draws the circle of radius slowest? In this section, we are going to be interested in parameterizations of curves wherethere is a one-to-one ratio between the parameter (the variable) and distance drawn(the arc length) from the start of the curve. Recall that if is a continuousvector-valued function where the curve drawn by is traversed once for , then the arclength of the curve from to is given by This is all good and well, but the integral could be quite difficult to compute. On the other hand, if were an arc lengthparameterization, this would be simple to compute, because then the arclength is in a one-to-one ratio with the variables. Hence Lets state this as adefinition.A curve traced out by a vector-valued function is parameterized by arc length if Such a parameterization is called an arc length parameterization. It is nice to work with functions parameterized by arc length, because computingthe arc length is easy. If is parameterized by arc length, then the lengthof when , is simply . No integral computations need to be done. Also weshould point out that is typically (though not necessarily) the name of thevariable when a function is parameterized by arc length, as often representsdistance. Suppose the curve below has an arc length parameterization given by . Compute: , , and Consider the following example:Let for . Show that is parameterized by arc length. Here we need to show that Well justcompute the right-hand side of the equation above and see what happens. Write withme, and so Now our integral becomes: Hence is parameterized by arc length.From your own experience and the work above, we think the next theorem should bequite sensible.A curve traced out by a continuously differentiable vector-valued function isparameterized by arc length if and only if .If we imagine our vector-valued function as giving the position of a particle, then thistheorem says that the path is parameterized by arc length exactly when the particleis moving at a speed of . Which of the following vector-valued functions are parameterized by arc length? Consider for . Find that makes this parameterized by arc length. Often given a curve one wishes to have an arc length parameterization of the curve.We proceed by discussing several special cases, and then by giving a generalmethod. Sometimes you have a vector-valued function that is merely a line in disguise. Howcould this be? Well consider the vector-valued function: This doesnt look very muchlike a line, for one thing it has the function in each component. On the other hand, ifwe look at , we see Ah, we can now factor a out of each component to get: this isa scalar-function times a constant vector. The fact that we can pull-outthe scalar function, and are left with a constant vector tells us that theline segment plotted by for is identical to the line segment plotted by: Which of the following are line segments in disguise? Once we identify a vector-valued function as a disguised line, we can rewriteit as and we have an arc length parameterization. Note, we need a unitvector to ensure that the magnitude of the derivative is one! Consider for .Parameterize this curve by arc length. If we think about we see that thevariable only appears in the expression as . This means as grows, it willgrow identically in each component of . Indeed a quick check with a graphwill show that a graph of produces the same graph as a graph of when .Ah, so this is a line in disguise! To parameterize a line by arc length you need towrite something like: So lets find two points on the line. Setting , we see that is onthe line. Setting we see that is also on the line. The unit vector that runs from to is: Thus as runs from to , draws the same curve as as runs from to. Give an arc length parameterization of for . for Try your hand at this one now: Give an arc length parameterization of for . Check the values of and. for Sometimes the curve we are given is a circle in disguise.Consider for . Parameterize this curve by arc length. Here, we should recognizethis curve a unit circle, being drawn in a counterclockwise fashion, starting(when) at the point . Ah! So an arc length parameterization is given by Consider for . Parameterize this curve by arc length. for The Moon travels in a orbit around the Earth that can be approximated by a circle.The distance from the Earth to the Moon is around thousand miles. Make thefollowing assumptions: The Earth will be at the origin. At the starting time, , the Moon will be at the point in the -plane. The Moon will travel in a counterclockwise direction around the Earth.Give a parameterization of the Moons orbit that will model the Moons position interms of , the distance traveled in thousands of miles. First compute thecircumference of the Moons orbit in thousands of miles: Now we write: While we are about to present a general method for finding representationsof functions parameterized by arc length, one must not overestimate itsstrength.Regardless, if you want an arc length parameterization of starting at here is theidea: (a) Compute (b) Now write and solve for . In this case you will have (c) The function will be parameterized by arc length.Try your hand at it.Parameterize for by arc length. First well compute the magnitude of . Write withme: So now: So now set : Now compute , in essence just solve for : Our curve, now parameterizedby arc length is Learning OutcomesDetermine the length of a particles path in space by using the arc-length function. We have seen how a vector-valued function describes a curve in either two or three dimensions. Recall Alternative Formulas for Curvature, which states that the formula for the arc length of a curve defined by the parametric functions [latex]x=x\,(t),\ y=t\,(t),\ t_{1}\,\leq\,t\,\leq\,t_{2}[/latex] is given by[latex]s=\displaystyle\int_{t_{1}}^{t^{2}}\ \sqrt{\big(x'\,(t)\big)^{2}+\big(y'\,(t)\big)^{2}}\,dt[/latex]In a similar fashion, if we define a smooth curve using a vector-valued function [latex]{\bf{r}}\,(t)=f\,(t)\,{\bf{I}}+g\,(t)\,{\bf{j}}[/latex], where [latex]a\,\leq\,t\,\leq\,b[/latex], the arc length is given by the formula[latex]s=\displaystyle\int_{a}^{b}\ \sqrt{\big(f'\,(t)\big)^{2}+\big(g'\,(t)\big)^{2}}\,dt[/latex]In three dimensions, if the vector-valued function is described by [latex]{\bf{r}}\,(t)=f\,(t)\,{\bf{I}}+g\,(t)\,{\bf{j}}+h\,(t)\,{\bf{k}}[/latex] over the same interval [latex]a\,\leq\,t\,\leq\,b[/latex], the arc length is given by[latex]s=\displaystyle\int_{a}^{b}\ \sqrt{\big(f'\,(t)\big)^{2}+\big(g'\,(t)\big)^{2}+\big(h'\,(t)\big)^{2}}\,dt[/latex]Plane curve: Given a smooth curve [latex]C[/latex] defined by the function [latex]{\bf{r}}\,(t)=g\,(t)\,{\bf{i}}+g\,(t)\,{\bf{j}}[/latex], where [latex]t[/latex] lies within the interval [latex][a,\ b][/latex], the arc length of [latex]C[/latex] over the interval is[latex]s=\displaystyle\int_{a}^{b}\ \sqrt{\big[f'\,(t)\big]^{2}+\big[g'\,(t)\big]^{2}}\,dt=\displaystyle\int_{a}^{b}\ \left\Vert{\bf{r}}'\,(t)\right\Vert\ dt[/latex]. Space curve: Given a smooth curve [latex]C[/latex] defined by the function [latex]{\bf{r}}\,(t)=f\,(t)\,{\bf{i}}+g\,(t)\,{\bf{j}}+h\,(t)\,{\bf{k}}[/latex], where [latex]t[/latex] lies within the interval [latex][a,\ b][/latex], the arc length of [latex]C[/latex] over the interval is[latex]s=\displaystyle\int_{a}^{b}\ \sqrt{\big[f'\,(t)\big]^{2}+\big[g'\,(t)\big]^{2}+\big[h'\,(t)\big]^{2}}\,dt=\displaystyle\int_{a}^{b}\ \left\Vert{\bf{r}}'\,(t)\right\Vert\ dt[/latex] The two formulas are very similar; they differ only in the fact that a space curve has three component functions instead of two. Note that the formulas are defined for smooth curves: curves where the vector-valued function [latex]{\bf{r}}\,(t)[/latex] is differentiable with a non-zero derivative.The smoothness condition guarantees that the curve has no cusps (or corners) that could make the formula problematic. Calculate the arc length for each of the following vector-valued functions: Calculate the arc length of the parameterized curve[latex]{\bf{r}}\,(t)=\langle{2t^{2}}+1,\ 2t^{2}-1,\ t^{3}\rangle,\ 0\,\leq\,t\,\leq\,3.[/latex] Watch the following video to see the worked solution to the above Try It You can view the transcript for CP 3.9 here (opens in new window).We now return to the helix introduced earlier in this chapter. A vector-valued function that describes a helix can be written in the form[latex]{\bf{r}}\,(t)=R\cos{\Big(\frac{2\pi\,N\,t}{h}\Big)}\,{\bf{i}}+R\sin{\Big(\frac{2\pi\,N\,t}{h}\Big)}\,{\bf{j}}+t\,{\bf{k}},\ 0\,\leq\,t\,\leq\,h,[/latex]where [latex]R[/latex]represents the radius of the helix, [latex]h[/latex]represents the height (distance between two consecutive turns), and the helix completes [latex]N[/latex]turns. Lets derive a formula for the arc length of this helix using Arc-Length Formulas. First of all,[latex]{\bf{r}}'\,(t)=-\frac{2\pi\,N\,R}{h}\sin{\Big(\frac{2\pi\,N\,t}{h}\Big)}\,{\bf{i}}+\frac{2\pi\,N\,R}{h}\cos{\Big(\frac{2\pi\,N\,t}{h}\Big)}\,{\bf{j}}+{\bf{k}}.[/latex]Therefore,[latex]\begin{array}{ccc}\hfill {s} &=\hfill&{\displaystyle\int_{a}^{b}\ \left\Vert{\bf{r}}'\,(t)\right\Vert\ dt}\hfill \\ \hfill & =\hfill & {\displaystyle\int_{0}^{h}\ \sqrt{\Bigg(-\frac{2\pi\,N\,R}{h}\sin{\Big(\frac{2\pi\,N\,t}{h}\Big)}\Bigg)^{2}+\Bigg(\frac{2\pi\,N\,R}{h}\cos{\Big(\frac{2\pi\,N\,t}{h}\Big)}\Bigg)^{2}+1^{2}}\,dt} \hfill \\ \hfill & =\hfill & {\displaystyle\int_{0}^{h}\ \sqrt{\frac{4{\pi}^{2}N^{2}R^{2}}{h^{2}}\,\Bigg(\sin{^{2}\Big(\frac{2\pi{N}t}{h}\Big)}+\cos{^{2}\Big(\frac{2\pi{N}t}{h}\Big)\Bigg)}+1}\,dt}\hfill \\ \hfill & =\hfill & {\displaystyle\int_{0}^{h}\ \sqrt{\frac{4{\pi}^{2}N^{2}R^{2}}{h^{2}}+1}\,dt}\hfill \\ \hfill & =\hfill & {\Big[t\,\sqrt{\frac{4{\pi}^{2}N^{2}R^{2}}{h^{2}}+1}\Big]^h_0}\hfill \\ \hfill & =\hfill & {h\,\sqrt{\frac{4{\pi}^{2}N^{2}R^{2}+h^{2}}{h^{2}}}}\hfill \\ \hfill & =\hfill & {\sqrt{4{\pi}^{2}N^{2}R^{2}+h^{2}}.}\hfill \\ \hfill \end{array}[/latex]This gives a formula for the length of a wire needed to form a helix with [latex]N[/latex]turns that has radius [latex]R[/latex]and height [latex]h[/latex].Arc-Length ParameterizationWe now have a formula for the arc length of a curve defined by a vector-valued function. Lets take this one step further and examine what an arc-length function is.If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time. The formula for the arc-length function follows directly from the formula for arc length:[latex]s\,(t)=\displaystyle\int_{a}^{t}\ \sqrt{\big(f'\,(u)\big)^{2}+\big(g'\,(u)\big)^{2}+\big(h'\,(u)\big)^{2}}\,du[/latex]If the curve is in two dimensions, then only two terms appear under the square root inside the integral. The reason for using the independent variable [latex]u[/latex]is to distinguish between time and the variable of integration. Since [latex]s(t)[/latex]measures distance traveled as a function of time, [latex]s'(t)[/latex]measures the speed of the particle at any given time. Since we have a formula for [latex]s(t)[/latex], we can differentiate both sides of the equation:[latex]\begin{array}{ccc}\hfill {s'\,(t)} &=\hfill&{\frac{d}{dt}\bigg[\displaystyle\int_{a}^{t}\ \sqrt{\big(f'\,(u)\big)^{2}+\big(g'\,(u)\big)^{2}+\big(h'\,(u)\big)^{2}}\,du\bigg]}\hfill \\ \hfill & =\hfill & {\frac{d}{dt}\bigg[\displaystyle\int_{a}^{t}\ \left\Vert{\bf{r}}'\,(u)\right\Vert\ du\bigg]} \hfill \\ \hfill & =\hfill & {\left\Vert{\bf{r}}'\,(t)\right\Vert\,.} \hfill \\ \hfill \end{array}[/latex]If we assume that [latex]{\bf{r}}\,(t)[/latex] defines a smooth curve, then the arc length is always increasing, so [latex]s'\,(t)\,>\,0[/latex] for [latex]t\,>\,a[/latex]. Last, if [latex]{\bf{r}}\,(t)[/latex] is a curve on which [latex]\left\Vert{\bf{r}}'\,(t)\right\Vert=1[/latex] for all [latex]t[/latex], then[latex]s\,(t)=\displaystyle\int_{a}^{t}\ \left\Vert{\bf{r}}'\,(u)\right\Vert\ du=\displaystyle\int_{a}^{t}\ 1\,du=t-a,[/latex]which means that [latex]t[/latex] represents the arc length as long as [latex]a=0[/latex]. Let [latex]{\bf{r}}\,(t)[/latex] describe a smooth curve for [latex]t\,\geq\,a[/latex]. Then the arc-length function is given by[latex]s\,(t)=\displaystyle\int_{a}^{t}\ \left\Vert{\bf{r}}'\,(u)\right\Vert\ du[/latex]. Furthermore, [latex]\frac{ds}{dt}=\left\Vert{\bf{r}}'\,(t)\right\Vert\,>\,0[/latex]. If [latex]\left\Vert{\bf{r}}'\,(t)\right\Vert=1[/latex] for all [latex]t\,\geq\,a[/latex], then the parameter [latex]t[/latex] represents the arc length from the starting point at [latex]t=a[/latex].A useful application of this theorem is to find an alternative parameterization of a given curve, called an arc-length parameterization. Recall that any vector-valued function can be reparameterized via a change of variables. For example, if we have a function [latex]{\bf{r}}\,(t)=\langle{3}\cos{t},\ 3\sin{t}\rangle,\ 0\,\leq\,t\,\leq\,2\pi[/latex]that parameterizes a circle of radius 3, we can change the parameter from [latex]t[/latex] to [latex]4t[/latex], obtaining a new parameterization [latex]{\bf{r}}\,(t)=\langle{3}\cos{4t},\ 3\sin{4t}\rangle[/latex]. The new parameterization still defines a circle of radius [latex]3[/latex], but now we need only use the values [latex]0\,\leq\,t\,\leq\,\frac{\pi}{2}[/latex] to traverse the circle once.Suppose that we find the arc-length function [latex]s(t)[/latex] and are able to solve this function for [latex]t[/latex] as a function of [latex]s[/latex]. We can then reparameterize the original function [latex]{\bf{r}}(t)[/latex] by substituting the expression for [latex]t[/latex] back into [latex]{\bf{r}}(t)[/latex]. The vector-valued function is now written in terms of the parameter [latex]s[/latex]. Since the variable [latex]s[/latex] represents the arc length, we call this an arc-length parameterization of the original function [latex]{\bf{r}}(t)[/latex]. One advantage of finding the arc-length parameterization is that the distance traveled along the curve starting from [latex]s=0[/latex] is now equal to the parameter [latex]s[/latex]. The arc-length parameterization also appears in the context of curvature (which we examine later in this section) and line integrals, which we study in the Introduction to Vector Calculus. Find the arc-length parameterization for each of the following curves: Find the arc-length function for the helix[latex]{\bf{r}}\,(t)=\langle{3}\cos{t},\ 3\sin{t},\ 4t\rangle{,}\ t\,\geq\,0.[/latex]Then, use the relationship between the arc length and the parameter [latex]t[/latex] to find an arc-length parameterization of [latex]{\bf{r}}(t)[/latex]. Watch the following video to see the worked solution to the above Try It You can view the transcript for CP 3.10 here (opens in new window). We may have a good notion of arc length even for some nonregular (or not even differentiable) curves; all we need (by definition) is that the curve $a\colon[0,b]\to \mathbb R^n$ is rectifiable, that is: $$ l(a):=\sup\ {\,d(a(0),a(t_1))+d(a(t_1),a(t_2))+\ldots+d(a(t_{n}),a(b))\mid 0